

# Study on direct pion emission in decay $D^{*+} \rightarrow D^+\pi$

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## Abstract

The QCD multipole expansion (QCDME) is based on the quantum field theory, so should be more reliable. However, on another aspect, it refers to the non-perturbative QCD, so that has a certain application range. Even though it successfully explains the data of transition among members of the  $\Upsilon$  ( $\psi$ ) family, as Eichten indicates, beyond the production threshold of mediate states it fails to meet data by several orders. In this work, by studying a simple decay mode  $D^* \rightarrow D + \pi^0$ , where a pion may be emitted before  $D^*$  transiting into  $D$ , we analyze the contribution of QCD multipole expansion. Whereas as the  $D\pi$  portal is open, the dominant contribution is an OZI allowed process where a light quark-pair is excited out from vacuum and its contribution can be evaluated by the  $^3P_0$  model. Since the direct pion emission is a process which is OZI suppressed and violates the isospin conservation, its contribution must be much smaller than the dominant one. By a careful calculation, we may quantitatively estimate how small the QCDME contribution should be and offer a quantitative interpretation for Eichten's statement.

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## I. INTRODUCTION

The QCD Multipole Expansion (QCDME) has been widely used to calculate transition rates among heavy quarkonia by emitting pions, [1, 2]. Since this theory refers to non-perturbative QCD, it has a limited application range, beyond this range the theory is no longer applicable. When the masses of the charmonia (bottononia) are sufficiently large beyond the production thresholds of  $D^{(*)}\bar{D}^{(*)}((B^{(*)}\bar{B}^{(*)}))$  which may become on-shell intermediate states, as Eichten et al. indicate, the decay widths evaluated in terms of the QCDME are smaller than the data by three orders[3]. In other words, the dominant modes of, say,  $\Upsilon(nS) \rightarrow \Upsilon(mS) + \pi^+\pi^-$  or  $\Upsilon(nS) \rightarrow \Upsilon(mS) + \pi^0$  can be realized via  $\Upsilon(nS) \rightarrow B^{(*)}\bar{B}^{(*)} \rightarrow \Upsilon(mS) + \pi^+\pi^-$ , which is usually referred as the final state interaction or re-scattering process. Even though the re-scattering process dominates the transition, the direct pion emission is still contributing and is evaluated in terms of the QCDME. It is interesting to theoretically estimate how small the contribution of the direct pion emission could be in comparison with the dominant one.

To serve the purpose, we adopt a simple decay mode to do the job, i.e. calculate the contribution of a direct  $\pi^0$  emission to the decay rate of  $D^* \rightarrow D + \pi^0$ .

For  $D^{*+} \rightarrow D^+\pi^0$ , the direct  $\pi^0$  emission is an OZI suppressed and moreover causes an isospin violation. The double suppression determines that the contribution from the direct pion emission must be small. In fact, unless other mechanisms are forbidden by some reasons, such as constraints of available phase space or other symmetries, the direct pion emission cannot make substantial contribution to the decay rates as Eichten et al. suggest. To quantitatively confirm Eichten's statement, we use both the  $^3P_0$  model and QCDME to calculate their contribution to the decay rate separately. Our numerical results show that the effective coupling constant  $g_{D^*D\pi}$  determined by QCDME is 60~70 times smaller than that obtained from quark pair creation (QPC).

After the introduction, in section II, we evaluate the contributions to the decay rate of the  $D^{*+} \rightarrow D^+\pi^0$  from both the quark pair creation (QPC) described by the  $^3P_0$  model and the direct pion emission described by the QCDME respectively in subsections II A and II B. The numerical results are presented following the formulations in the section and comparisons with the corresponding experimental data are made. In the final section we will discuss the framework in some details and then draw our conclusion.

## II. $D^{*+} \rightarrow D^+\pi^0$ DECAYS

### A. The quark pair creation model and its application to $\pi^0$ radiation

In the framework of the QPC model[4–21], the decay  $D^{*+} \rightarrow D^+\pi^0$  occurs via a quark-antiquark pair creation from the vacuum. It is an Okubo-Zweig-Iizuka (OZI) allowed process. The decay mechanism is displayed in Fig.1 graphically. The picture is that many soft gluons are emitted from the quark and anti-quark legs which then annihilate into a quark-antiquark pair. Equivalently, the physics scenario can be described as that a quark pair is excited out from vacuum. The  $^3P_0$  model has been widely applied to calculate such hadronic strong decays.

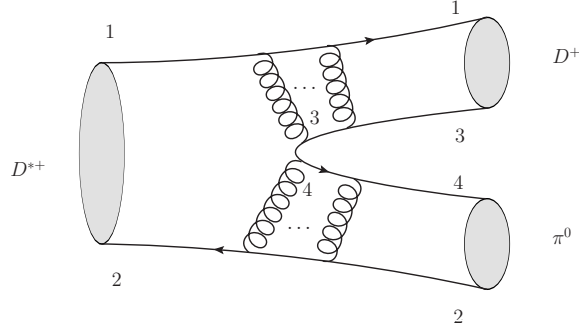


FIG. 1: The quark-pair creation from vacuum serves as the decay mechanism for  $D^{*+} \rightarrow D^+\pi$ .

For readers' convenience, we collect the relevant formulations about the calculation in terms of the  $^3P_0$  model in the appendix. The transition operator for the quark pair creation reads

$$T = -3\gamma \sum_m \langle 1 \ m; 1 \ -m | 0 \ 0 \rangle \int d\mathbf{k}_3 d\mathbf{k}_4 \delta^3(\mathbf{k}_3 + \mathbf{k}_4) \mathcal{Y}_{1m} \left( \frac{\mathbf{k}_3 - \mathbf{k}_4}{2} \right) \times \chi_{1,-m}^{34} \varphi_0^{34} \omega_0^{34} d_{3i}^\dagger(\mathbf{k}_3) b_{4j}^\dagger(\mathbf{k}_4), \quad (1)$$

and the hadronic matrix element is determined as

$$\langle D^+\pi^0 | S | D^{*+} \rangle = I - i2\pi\delta(E_{\text{final}} - E_{\text{initial}}) \langle D^+\pi^0 | T | D^{*+} \rangle. \quad (2)$$

In Eq. 1,  $i$  and  $j$  are the SU(3)-color indices of the created quark and anti-quark.  $\varphi_0^{34} = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$  and  $\omega_0^{34} = \delta_{ij}$  are for flavor and color singlets, respectively.  $\chi_{1,-m}^{34}$  is the

spin wave function.  $\mathcal{Y}_{\ell m}(\mathbf{k}) \equiv |\mathbf{k}|^\ell Y_{\ell m}(\theta_k, \phi_k)$  is the  $\ell$ th solid harmonic polynomial.  $\gamma$  is a dimensionless constant which denotes the strength of quark pair creation from vacuum and is fixed by fitting data. Following Ref.[22], we take  $\gamma = 13.4$  in this work. For Eq. 2, the explicit expressions for the wave function of a meson and the hadronic matrix elements are presented in the appendix.

The helicity amplitude  $\mathcal{M}^{M_{J_{D^{*+}}} M_{J_{D^+}} M_{J_{\pi^0}}}$  of this process can be extracted from the relation  $\langle D^+ \pi^0 | S | D^{*+} \rangle = \delta^3(\mathbf{K}_{D^+} + \mathbf{K}_{\pi^0} - \mathbf{K}_{D^{*+}}) \mathcal{M}^{M_{J_{D^{*+}}} M_{J_{D^+}} M_{J_{\pi^0}}}$ . Then, the decay width corresponding to the process is written in terms of the helicity amplitude as

$$\Gamma = \pi^2 \frac{|\mathbf{K}|}{M_{D^{*+}}^2} \frac{1}{2J_{D^{*+}} + 1} \sum_{M_{J_{D^{*+}}}, M_{J_{D^+}}, M_{J_{\pi^0}}} \left| \mathcal{M}^{M_{J_{D^{*+}}} M_{J_{D^+}} M_{J_{\pi^0}}} \right|^2,$$

where we take  $\mathbf{K}_{D^+} = -\mathbf{K}_{\pi^0} = \mathbf{K}$  in the center of the mass frame of  $D^{*+}$ .

Numerically, we take a typical  $R$  value for  $D$  meson from Ref. [17] as  $2.3 \text{ GeV}^{-1}$  and  $R = 2.1 \text{ GeV}^{-1}$  for  $\pi^0$  from Ref. [22]. With these parameter setup, the decay width of  $D^{*+} \rightarrow D^+ \pi^0$  can be easily obtained as  $21.9 \text{ keV}$ . Experimentally, the decay width of this mode is  $29.5_{-7.2}^{+7.3} \text{ keV}$ [23]. The consistency of the numerical results evaluated in terms of the  $^3P_0$  model with data indicates that the theoretical framework is well established and applicable to describe such processes.

On the other hand, based on the heavy quark effective theory(HQET), one can extract the effective coupling constant of  $D^* D \pi$  from the afore calculated decay width which might offer significant information for application of an effective theory. Following Ref.[24], the related effective Lagrangian can be written as

$$\mathcal{L} = -\frac{2g_{D^* D \pi}}{f_\pi} D_\mu^* \partial^\mu \frac{\phi_\pi}{\sqrt{2}} D^\dagger + h.c. \quad (3)$$

Then we can get the decay width as

$$\Gamma(D^{*+} \rightarrow D^+ \pi^0) = \frac{1}{2m_{D^*}} \frac{4\pi|\mathbf{k}|}{(2\pi)^2 4m_{D^*}} \frac{|\mathcal{T}|^2}{3}, \quad (4)$$

with  $|\mathbf{k}| = \frac{1}{2m_{D^*}} [(m_{D^*}^2 - (m_\pi + m_D)^2)(m_{D^*}^2 - (m_\pi - m_D)^2)]^{1/2}$ . The transition amplitude of  $D^{*+} \rightarrow D^+ \pi^0$  is[24]

$$\mathcal{T}(D^{*+} \rightarrow D^+ \pi^0) = g_{D^* D \pi} \frac{1}{\sqrt{2}} \frac{2m_D}{f_\pi} \mathbf{k} \cdot \boldsymbol{\epsilon}, \quad (5)$$

here  $\boldsymbol{\epsilon}$  is the polarization vector of  $D^*$ . From equation(4) we obtain  $g_{D^* D \pi} = 0.51$ .

## B. The QCDME and evaluating contribution of direct $\pi^0$ emission to the decay width

It is noted that the pion can be directly emitted before  $D^*$  transits into  $D$ , thus the amplitude, in principle, should be added to the process depicted in above subsection and interferes with it. Just by the qualitative analysis, the one-pion emission is an OZI suppressed process and moreover, it violates isospin conservation, therefore must be much small compared to the vacuum creation.

It is obviously interesting to investigate such an effect in other processes, i.e. as long as the one-pion emission is not a leading term, how small it would be compared with the leading ones. Below, we will quantitatively investigate the direct one-pion emission in  $D^* \rightarrow D + \pi^0$ .

The corresponding diagrams for the direct pion emission in  $D^{*+} \rightarrow D^+ \pi^0$  for which the QCDME is responsible are shown in Fig.2.

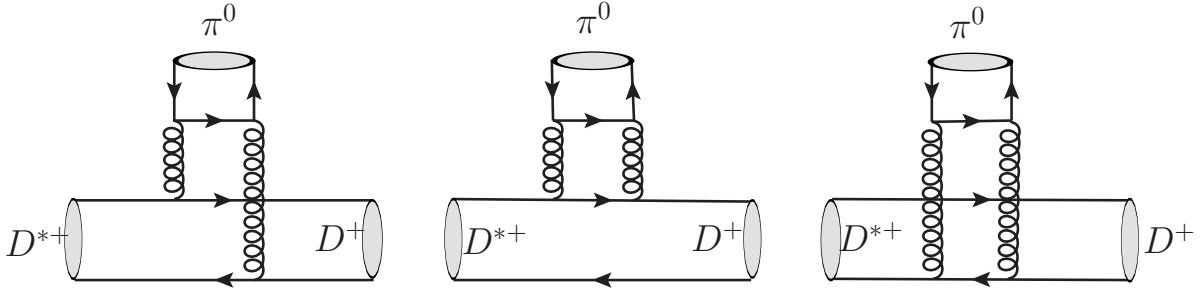


FIG. 2: The QCDME diagrams responsible for pion emission in the process  $D^{*+} \rightarrow D^+ \pi^0$ .

The readers should note that Fig.2 is just obtained by distorting Fig.1, as shown in Fig.3.

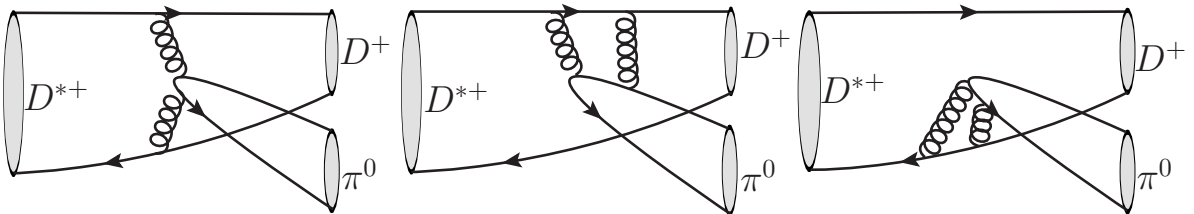


FIG. 3: Distortion of the  $^3P_0$  decay mechanism for  $D^{*+} \rightarrow D^+ \pi$  into an OZI suppressed process.

Here we only draw two gluon field lines, but as well understood, in the scenario of QCDME the lines correspond to a field of  $E_n$  mode or  $M_n$  one which are by no means free gluons and

the line is also not corresponding to a single-gluon propagator. Thus the line indeed denote a collection of many soft gluons just as shown in Fig.1.

Now let us calculate the rate contributed by the processes shown in Fig.2 in the framework of QCDME.

The process of directly emitting a soft  $\pi^0$  from  $D^*$  in decay  $D^* \rightarrow D + \pi^0$  is dominated by an E1-M2 coalesce transition. This is an OZI suppressed process and violates isospin conservation. The transition amplitude is[25]

$$\mathcal{M}_{E1M2} = i \frac{g_E g_M}{12m} \sum_{NL} \left( \frac{\langle \Phi_F | x_i | NL \rangle \langle NL | S_j x_k | \Phi_I \rangle}{M_I - E_{NL}} + \frac{\langle \Phi_F | S_j x_k | NL \rangle \langle NL | x_i | \Phi_I \rangle}{M_I - E_{NL}} \right) \langle \pi | E_i^a \partial_k B_j^a | 0 \rangle, \quad (6)$$

where  $S$  operator acting on the total spin of the heavy-quark and light-anti-quark system,  $N$  and  $L$  are the principal quantum number and the orbital angular momentum of the intermediate hybrid state,  $M_I$  and  $E_{NL}$  are the mass of the initial meson  $D^*$  and the energy eigenvalues of the hybrid state,  $m$  is the energy scale of the M2 transition and we set it to be  $m_c$  and  $\frac{m_c}{2}$  in our numerical computations. The amplitude reduces into[26, 27]

$$\mathcal{M}_{E1M2} = i \frac{g_E g_M}{18m} \sum_{NL} \frac{\int R_F(r) r R_{NL}^*(r) r^2 dr \int R_{NL}^*(r') r' R_I(r') r'^2 dr'}{M_I - E_{NL}} \epsilon_k \langle \pi | E_l^a \partial_l B_k^a | 0 \rangle, \quad (7)$$

where  $\epsilon$  is the polarization vector of  $D^*$ ,  $R_I$ ,  $R_F$  and  $R_{NL}$  are the radial wave functions of the initial, final and intermediate hybrid state, respectively.

The radial wave functions are calculated via solving the relativistic Schrödinger equation [28]. The potentials for the initial and final  $D^{(*)}$  mesons are taken from Ref.([28]) and the potential for the intermediate hybrid states is taken from Ref.([29]).

The matrix element  $\langle \pi | g_E g_M E_l^a \partial_l B_k^a | 0 \rangle$  is of the form[26, 27]

$$\langle \pi | g_E g_M E_l^a \partial_l B_k^a | 0 \rangle = \frac{1}{12} K k \frac{g_E g_M}{\alpha_s} \frac{4\pi}{\sqrt{2}} \frac{m_d - m_u}{m_d + m_u} f_\pi m_\pi^2, \quad (8)$$

where  $g_E$  and  $g_M$  are the coupling constants for the color electric field and color magnetic field,  $\mathbf{k}$  is the momentum of  $\pi^0$ .

In order to compare the results with the effective coupling constant  $g_{D^* D \pi}$  obtained by using the QPC model, the transition amplitude of  $D^{*+} \rightarrow D^+ \pi^0$  can be rewritten as[24]

$$\mathcal{M}(D^{*+} \rightarrow D^+ \pi^0) = g_{D^* D \pi}^{(ME)} \frac{2m_D}{f_\pi} \mathbf{k} \cdot \epsilon. \quad (9)$$

$g_{D^* D \pi}^{(ME)}$  is the effective coupling constant obtained by means of QCDME, it is then

$$g_{D^* D \pi}^{(ME)} = \frac{1}{18m} f_{1110} \frac{g_E g_M}{\alpha_s} \frac{\pi}{3\sqrt{2}} \frac{m_u - m_d}{m_u + m_d} f_\pi m_\pi^2 \sqrt{2m_{D^*}} \sqrt{2m_D} \frac{f_\pi}{2m_D}, \quad (10)$$

with

$$f_{1110}^{111} = \sum_{NL} \frac{\int R_F(r) r R_{NL}^*(r) r^2 dr \int R_{NL}^*(r') r' R_I(r') r'^2 dr'}{M_I - E_{NL}}. \quad (11)$$

For our numerical analysis, the input parameters are taken from Ref.[23], here  $m_D = 1.869$  GeV,  $m_{D^*} = 2.010$  GeV,  $m_\pi = 0.135$  GeV,  $m_c = 1.800$  GeV,  $m_u = 0.3$  GeV,  $\frac{m_d - m_u}{m_d + m_u} = \frac{1}{3}$ ,  $f_{B^*} = 0.230$  GeV,  $f_K = 0.160$  GeV,  $F_\pi = 0.093$  GeV and  $f_\pi = \sqrt{2}F_\pi$ .  $\alpha_s = 0.31$  for  $\sqrt{s} = 2.010$  GeV. Following Ref. [1, 2, 26, 27] we set  $\alpha_E = \frac{g_E^2}{4\pi}$ ,  $\alpha_M = \frac{g_M^2}{4\pi}$  with  $\alpha_E = 0.6$  and for a possible error range, according to the literature, we let  $\alpha_M$  vary from  $\alpha_E$  to  $10\alpha_E$ . The constants  $f_{1110}^{111}$  and effective coupling constant  $g$  obtained in terms of QCDME are listed in Tab.I.

	$\alpha_M = \alpha_E$	$\alpha_M = 3\alpha_E$	$\alpha_M = 10\alpha_E$	$\alpha_M = 30\alpha_E$
$f_{1110}^{111}$	5.677	9.833	17.952	31.094
$g_{D^*D\pi}(QCDME)$	0.00145	0.00251	0.00459	0.00794

TABLE I: coupling constant  $g_{D^*D\pi}^{(ME)}$  and  $f_{1110}^{111}$  (in units of  $\text{GeV}^{-3}$ ), and we set  $\alpha_M$  to be  $\alpha_E$ ,  $3\alpha_E$ ,  $10\alpha_E$ , and  $30\alpha_E$  separately, and the value of  $m$  is set to  $\frac{m_c}{2}$ .

In order to explore possible validity ranges of QCDME, we extend the maximum value of  $\alpha_M$  to  $30\alpha_E$ . One can see that when  $\alpha_M$  takes the value  $30\alpha_E$ ,  $g_{D^*D\pi}^{(ME)}$  reaches 0.00794. With possible errors, this result is 60 times smaller than that obtained by the QPC model[23, 30].

	$\alpha_M = \alpha_E$	$\alpha_M = 3\alpha_E$	$\alpha_M = 10\alpha_E$	$\alpha_M = 30\alpha_E$
$\Gamma(D^{*+} \rightarrow D^+\pi^0)$ with $m = \frac{m_c}{2}$	$4.43 \times 10^{-5}$	$5.31 \times 10^{-4}$	$4.44 \times 10^{-4}$	$5.31 \times 10^{-3}$
$\Gamma(D^{*+} \rightarrow D^+\pi^0)$ with $m = m_c$	$1.77 \times 10^{-4}$	$1.33 \times 10^{-4}$	$1.78 \times 10^{-3}$	$1.33 \times 10^{-3}$

TABLE II: Decay width of  $\Gamma(D^{*+} \rightarrow D^+\pi^0)$ (in unit of keV) from QCDME contribution, for  $\alpha_M$  we takes  $\alpha_E$ ,  $3\alpha_E$ ,  $10\alpha_E$ , and  $30\alpha_E$  respectively.

In Tab.II we also list decay width  $\Gamma(D^{*+} \rightarrow D^+\pi^0)$  from QCDME contribution, from this table we can see when  $\alpha_M$  takes the maximum value  $30\alpha_E$ , the decay width of  $\Gamma(D^{*+} \rightarrow$

$D^+\pi^0$ ) from QCDME contribution is  $5.31 \times 10^{-3}\text{keV}$ . This result is more than three orders smaller than the result obtained by using the QPC model( $21.9\text{keV}$ ).

### III. CONCLUSION AND DISCUSSION

Our numerical results for decay mode  $D^* \rightarrow D\pi^0$  confirm that the direct pion emission is not the leading term and the contributions determined by the QCDME must be much smaller than that induced by other mechanisms.

Let us try to understand the smallness of the contribution from direct pion emission, namely using data to convince that our estimate is reasonable. There are two suppression factors: the one-pion emission violates the isospin conservation and QCDME is an OZI suppressed mechanism. Comparing with the vacuum quark pair creation, it must be small and we see that its contribution to the decay width ranges about  $10^{-4} \sim 10^{-5} \text{ keV}$ . As we know, the decay  $\Psi(2S) \rightarrow J/\Psi + \pi^0$  is also an isospin violation and OZI suppressed process, its branching ratio is  $1.27 \times 10^{-3}$ [23], slightly larger than our estimate for  $D^* \rightarrow D\pi^0$ . That further suppression factor is coming from the fact that the transition  $\Psi(2S) \rightarrow J/\Psi + \pi^0$  is an  $S$ -wave process, whereas  $D^{*+} \rightarrow D^+\pi^0$  is a  $P$ -wave one whose decay width is proportional to the three-momentum  $\mathbf{k}$  which is small, so the decay width  $\Gamma(D^{*+} \rightarrow D^+\pi^0)$  suffers from a  $P$ -wave suppression.

We may also look at  $\psi(2S)$  as an example for a direct  $\pi$  emission.  $\psi(2S) \rightarrow \pi^0 + h_c(1P)$  can only occur via a direct pion emission, so is completely determined by the QCDME mechanism, and its partial width is about  $0.26 \text{ keV}$ . Since it is an  $S$ -wave process, it does not suffer from the  $P$ -wave suppression. In  $\psi(2S)$  decays, the mode  $\psi(2S) \rightarrow \eta_c + \pi^0$  is not seen, but  $\psi(2S) \rightarrow \eta_c + \pi^+\pi^-\pi^0$  has been measured, and its branching ratio is not too small ( $< 1.0 \times 10^{-3}$ ), that is because the direct emission of three pions does not violate the isospin conservation.

Equivalently, the QCDME can be replaced by the chiral perturbation theory, for example the transition of  $\Upsilon(nS) \rightarrow \Upsilon(mS) + \pi^+\pi^-$  was studied in terms of the chiral theory[31].

Moreover, we have extended our mechanism to study the non-resonant three-body decays of  $B$  where the weak interaction gets involved. The contribution has been studied by Cheng et al. in terms of the chiral perturbation theory and we have re-done the evaluation by means of QCDME. We will present the relevant results in our next work.



As a conclusion, we confirm the validity of the QCDEM and determine its application region. It is indicated that since the framework applies only to the direct pion emission, if there are other mechanisms to contribute, such as the quark-pair creation from vacuum to  $D^* \rightarrow D + \pi^0$ ; or  $\Upsilon(5S) \rightarrow B\bar{B}^* \rightarrow \Upsilon(1S) + \pi^+\pi^-$  where an intermediate state of  $B\bar{B}^*$  portal is open, i.e the available energy is above the production threshold of  $B\bar{B}^*$ , the direct emission induced by the QCDEM is no longer the leading term and can only contribute tiny fraction.

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### Appendix: Some formulae

In the  $^3P_0$  model, for a two-body decay process  $A \rightarrow BC$ , the total wave function of a meson can be written as

$$\begin{aligned}
& \left| A(n_A {}^{2S_A+1}L_A J_A M_{J_A}) (\mathbf{K}_A) \right\rangle \\
&= \sqrt{2E_A} \sum_{M_{L_A}, M_{S_A}} \langle L_A M_{L_A} S_A M_{S_A} | J_A M_{J_A} \rangle \\
&\quad \times \int d\mathbf{k}_1 d\mathbf{k}_2 \delta^3(\mathbf{K}_A - \mathbf{k}_1 - \mathbf{k}_2) \Psi_{n_A L_A M_{L_A}}(\mathbf{k}_1, \mathbf{k}_2) \\
&\quad \times \chi_{S_A M_{S_A}}^{12} \varphi_A^{12} \omega_A^{12} | q_1(\mathbf{k}_1) \bar{q}_2(\mathbf{k}_2) \rangle, \tag{A.1}
\end{aligned}$$

which satisfies the normalization condition:  $\langle A(\mathbf{K}_A) | A(\mathbf{K}'_A) \rangle = 2E_A \delta^3(\mathbf{K}_A - \mathbf{K}'_A)$ . The subscripts 1 and 2 refer to the quark and anti-quark within meson  $A$ .  $\mathbf{K}_A$  is the momentum of the meson  $A$ .  $\chi_{S_A M_{S_A}}^{12}, \varphi_A^{12}, \omega_A^{12}$  are the spin, flavor and color parts respectively.  $\Psi_{n_A L_A M_{L_A}}(\mathbf{k}_1, \mathbf{k}_2)$  is the spatial part of a meson wavefunction in the momentum representation. For the concerned mesons are in the ground states, the simple harmonic oscillator (HO) wavefunction is employed which reads as

$$\Psi_{00}(\mathbf{k}) = \frac{1}{\pi^{3/4}} R^{3/2} \exp\left(-\frac{R^2 \mathbf{k}^2}{2}\right), \tag{A.2}$$

where  $\mathbf{k} = (m_2\mathbf{k}_1 - m_1\mathbf{k}_2)/(m_1 + m_2)$  is the relative momentum between the quark 1 (with mass  $m_1$ ) and the anti-quark 2 (with mass  $m_2$ ) within a meson. The  $R$  value is fixed by fitting experimental data.

The explicit matrix element is depicted as

$$\begin{aligned}
\langle BC|T|A\rangle &= \sqrt{8E_A E_B E_C} \gamma \sum_{\substack{M_{L_A}, M_{S_A}, \\ M_{L_B}, M_{S_B}, \\ M_{L_C}, M_{S_C}, m}} \langle 1\ m; 1\ -m | 0\ 0 \rangle \\
&\times \langle L_A M_{L_A} S_A M_{S_A} | J_A M_{J_A} \rangle \langle L_B M_{L_B} S_B M_{S_B} | J_B M_{J_B} \rangle \\
&\times \langle L_C M_{L_C} S_C M_{S_C} | J_C M_{J_C} \rangle \langle \varphi_B^{13} \varphi_C^{24} | \varphi_A^{12} \varphi_0^{34} \rangle \\
&\times \langle \chi_{S_B M_{S_B}}^{13} \chi_{S_C M_{S_C}}^{24} | \chi_{S_A M_{S_A}}^{12} \chi_{1-m}^{34} \rangle I_{M_{L_B}, M_{L_C}}^{M_{L_A}, m}(\mathbf{K}),
\end{aligned}$$

where  $I_{M_{L_B}, M_{L_C}}^{M_{L_A}, m}(\mathbf{K})$  is a spatial integral, reading as

$$\begin{aligned}
I_{M_{L_B}, M_{L_C}}^{M_{L_A}, m}(\mathbf{K}) &= \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \\
&\times \delta^3(\mathbf{k}_3 + \mathbf{k}_4) \delta^2(\mathbf{K}_B - \mathbf{k}_1 - \mathbf{k}_3) \delta^3(\mathbf{K}_C - \mathbf{k}_2 - \mathbf{k}_4) \\
&\times \Psi_{n_B L_B M_{L_B}}^*(\mathbf{k}_1, \mathbf{k}_3) \Psi_{n_C L_C M_{L_C}}^*(\mathbf{k}_2, \mathbf{k}_4) \\
&\times \Psi_{n_A L_A M_{L_A}}(\mathbf{k}_1, \mathbf{k}_2) \mathcal{Y}_{1m}\left(\frac{\mathbf{k}_3 - \mathbf{k}_4}{2}\right).
\end{aligned} \tag{A.3}$$

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